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On Oppenheim-type conjecture for systems of quadratic forms*

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Abstract

Let Q_i, $i = 1, \dots, t$, be real nondegenerate indefinite quadratic forms in d variables. We investigate under what conditions the closure of the set $\{(Q_1(\bar{x}), \dots, Q_t(\bar{x}))\colon \bar{x} \in \mathbb{Z}^d - \{\bar{0}\}\}$ contains $(0, \dots, 0)$. As a corollary, we deduce several results on the magnitude of the set Δ of $g \in \mathrm{GL}(d, \mathbb{R})$ such that the closure of the set $\{(Q_1(g\bar{x}), \dots, Q_t(g\bar{x})) : \bar{x} \in \mathbb{Z}^d - \{\bar{0}\}\}$ contains $(0, \dots, 0)$. Special cases are described when depending on the mutual position of the hypersurfaces $\{Q_i = 0\}$, $i = 1, \dots, t$, the set Δ has full Haar measure or measure zero and Hausdorff dimension $d^2 - \frac{d-2}{2}$.

1 Introduction

It was conjectured by Oppenheim that if

$$Q(\bar{x}) = \sum_{i,j} a_{ij} x_i x_j, \quad \bar{x} = (x_1, \dots, x_d),$$

is a real nondegenerate indefinite quadratic form in d variables, $d \geq 5$, which is not proportional to a rational form, then

$$\forall \varepsilon > 0 \;\; \exists \bar{x} \in \mathbb{Z}^d - \{\bar{0}\}\colon \; |Q(\bar{x})| < \varepsilon. \tag{1}$$

This conjecture was proved by Margulis [Ma89]. In fact, he proved that if Q is a real nondegenerate indefinite quadratic form in d variables, $d \geq 3$, which is not proportional to a rational form, then

$$\forall \varepsilon > 0 \;\; \exists \bar{x} \in \mathbb{Z}^d - \{\bar{0}\}\colon \; 0 < |Q(\bar{x})| < \varepsilon.$$

See [Ma97] for an up-to-date survey. It is also known that (1) fails for some indefinite irrational forms in 2 variables. For example, (1) fails for $Q = x_1^2 - \alpha^2 x_2^2$ with a badly

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approximable number α (e.g. $\alpha = 1+\sqrt{2}$). Recall that a number α is called **badly approximable** if there exists $c>0$ such that for all integers p and $q>0$,

$$|q\alpha - p| \geq \frac{c}{q}.$$

In this paper we study similar questions for systems ($Q_j\colon j=1,\dots,t$) of quadratic forms. Namely, we would like to know when the following statement is true:

$$\forall \varepsilon > 0 \;\; \exists \bar{x} \in \mathbb{Z}^d - \{\bar{0}\}\colon \;\; \max_{j=1,\dots,t} |Q_j(\bar{x})| < \varepsilon. \tag{2}$$

We expect that an analog of the Oppenheim conjecture holds for systems of quadratic forms (see Conjecture 13 below). However, the proof of this conjecture seems to be beyond the reach of available methods.

One can give a convenient characterization of property (2) for systems quadratic forms in two variables: (2) holds for nondegenerate indefinite quadratic forms

$$Q_j(x_1,x_2) = (a_j x_1 + b_j x_2)(c_j x_1 + d_j x_2), \;\; j=1,\dots,t,$$

iff the set

$$\bigcap_{j=1}^{t} \left\{ \frac{a_j}{b_j}, \frac{c_j}{d_j} \right\}$$

contains a number which is not badly approximable (we consider ∞ to be not badly approximable). In the case of one quadratic form, this was observed by Dani [Da00]. Theorem 1 below is an analog of this fact in dimension $d>2$.

Validity of (2) depends on the common position of the hypersurfaces $\{Q_j = 0\}$, $j=1,\dots,t$. For example, it is easy to see that if these hypersurfaces intersect only at the origin, then (2) fails. We investigate the case when the intersection of the hypersurfaces $\{Q_j=0\}$, $j=1,\dots,t$, is still relatively small: they intersect transversally in a strong sense. To formulate our result, we recall that a vector $\bar{\alpha} = (\alpha_1,\dots,\alpha_d) \in \mathbb{R}^d$ is called **well approximable** of order r ($r>0$) if there exist $\bar{p}_n \in \mathbb{Z}^d$ and $q_n > 0$ such that

$$q_n^r \cdot \|q_n\bar{\alpha} - \bar{p}_n\| \to 0 \;\text{ as }\; n\to\infty. \tag{3}$$

Theorem 1 *Let Q_j, $j=1,\dots,t$, be nondegenerate indefinite quadratic forms in d variables, $d\geq 3$. Suppose that the intersection of the hypersurfaces $\{Q_j=0\}$, $j=1,\dots,t$, is not $\{\bar{0}\}$, and at every point of intersection, which is different from $\bar{0}$, the space spanned by the normal vectors to the hypersurfaces has dimension $d-1$. Then (2) holds iff the intersection of the hypersurfaces $\{Q_j=0\}$, $j=1,\dots,t$, contains a vector $\bar{v}\in\mathbb{R}^d$ such that for some $j=1,\dots,t$, $v_j\neq 0$, and the vector $\left(\frac{v_i}{v_j}\colon i\neq j\right)\in\mathbb{R}^{d-1}$ is well approximable of order one.*

We also study the case when zero hypersurfaces of the quadratic forms have a common tangent plane at a point of intersection. Choose and fix a line along which the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$, have a common tangent plane. Let $\bar{f}_1$ be the direction vector of this line. We choose a basis $\{\bar{f}_i\colon i = 1, \ldots, d\}$ such that the vector $\bar{f}_2$ is outside of the tangent plane, and $\bar{f}_i$, $i = 3, \ldots, d$, are in the tangent plane. Define $a_t \in \mathrm{SL}(d, \mathbb{R})$ by

$$a_t\bar{f}_1 = e^{-t}\bar{f}_1, \quad a_t\bar{f}_2 = e^{t}\bar{f}_2, \quad a_t\bar{f}_i = \bar{f}_i,\ i = 3, \ldots, d. \tag{4}$$

The following theorem establishes a connection between properties of the semiorbit of a_t, $t > 0$, in $\mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ and Diophantine condition (2). First results of this type were discovered by Dani [Da85, Da86].

Theorem 2 *Let Q_j, $j = 1, \ldots, t$, be nondegenerate indefinite quadratic forms in d variables, $d \geq 3$. Suppose that the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$, have a common tangent plane at a point of intersection, which is different from $\bar{0}$. If the semiorbit*

$$\{a_t\mathrm{SL}(d, \mathbb{Z})\colon t > 0\} \subseteq \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z}),$$

where a_t is defined in (4), is not relatively compact, then (2) holds. Moreover, if for some $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, t$, the quadratic form $\sum_j \alpha_j Q_j$ is definite of rank $d - 1$, then (2) implies that the semiorbit $\{a_t\mathrm{SL}(d, \mathbb{Z})\colon t > 0\}$ is not relatively compact in $\mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$.

Remark Since the choice of the vectors $\bar{f}_i$, $i = 2, \ldots, d$, is not unique, the transformation a_t is not uniquely defined. Nonetheless, one can check that the property that the positive semiorbit of a_t is bounded is independent of a chosen basis.

Theorems 1 and 2 allow us to investigate how often (2) holds in the space of systems of quadratic forms. For quadratic forms Q_j, $j = 1, \ldots, t$ in d variables, we consider a parametric family

$$(Q_j^g\colon j = 1, \ldots, t), \ \ g \in \mathrm{GL}(d, \mathbb{R}),$$

where $Q_j^g(\bar{x}) \overset{def}{=} Q(g\bar{x})$, and denote by $\Delta(Q_1, \ldots, Q_t)$ the set of $g \in \mathrm{GL}(d, \mathbb{R})$ such that $(Q_j^g\colon j = 1, \ldots, t)$ satisfies (2).

Corollary 3 *Let Q_j, $j = 1, \ldots, t$, be nondegenerate indefinite quadratic forms in d variables, $d \geq 3$, and $\Delta = \Delta(Q_1, \ldots, Q_t)$.*

(i) If the intersection of the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$, is not $\{\bar{0}\}$, and at every point of intersection, which is different from $\bar{0}$, the space spanned by the normal vectors to the hypersurfaces has dimension $d - 1$, then Δ has measure zero and Hausdorff dimension $d^2 - \frac{d-2}{2}$.

(ii) If the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$, have a common tangent plane at a point of intersection, which is different from $\bar{0}$, then Δ has full Haar measure in $\mathrm{GL}(d, \mathbb{R})$ (i.e. its complement has measure zero). Moreover, if for some $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, t$, the quadratic form $\sum_j \alpha_j Q_j$ is definite of rank $d-1$, then the complement of Δ in $\mathrm{GL}(d, \mathbb{R})$ has Hausdorff dimension d^2.

Recall that the **radical** $\mathrm{Rad}(Q)$ of a quadratic form Q, which is defined on a vector space S, is the subspace of vectors in S that are orthogonal to S with respect to Q.

We prove the following result that complements Corollary 3(ii):

Theorem 4 *Let Q_j, $j = 1, \ldots, t$, be nondegenerate indefinite quadratic forms in d variables, $d \geq 3$, and $\Delta = \Delta(Q_1, \ldots, Q_t)$. Suppose that for some $\beta_j \in \mathbb{R}$, $j = 2, \ldots, d$,*

$$V \stackrel{def}{=} \bigcap_{j=2,\ldots,d} \mathrm{Rad}(Q_j - \beta_j Q_1)$$

has dimension at least 2, and V contains a vector $\bar{v} \neq \bar{0}$ such that $Q_1(\bar{v}) = 0$.

(a) Unless $\dim V = 2$ and the quadratic form $Q_1|_V$ is nondegenerate, the complement of Δ in $\mathrm{GL}(d, \mathbb{R})$ is contained in a countable union of proper submanifolds of dimension at most $d^2 - d + 1$.

(b) If $\dim V = 2$, the quadratic form $Q_1|_V$ is nondegenerate, and for some $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, t$, the quadratic form $\sum_j \alpha_j Q_j$ is definite of rank $d-2$, then the complement of Δ in $\mathrm{GL}(d, \mathbb{R})$ has Hausdorff dimension d^2.

Note that the conditions of Theorem 4 imply that the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$, have a common tangent plane at $\bar{v}$. In particular, it follows from Corollary 3(ii) that the complement of Δ has measure 0.

The simplest example for Theorem 4(a) is provided by the system of quadratic forms

$$Q_j(\bar{x}) = x_1^2 + x_2 x_3 + a_j x_3^2, \quad j = 1, \ldots, t.$$

It follows from the result of Dani and Margulis [DM90] that the complement of $\Delta(Q_1, Q_2)$ is a countable union of proper submanifolds of $\mathrm{GL}(3, \mathbb{R})$.

It is easy to refine Corollary 3 and Theorem 4 as follows. Denote by Δ' the set of $g \in \mathrm{GL}(d, \mathbb{R})$ such that

$$\forall \varepsilon > 0 \ \ \exists \bar{x} \in \mathbb{Z}^d - \{\bar{0}\}\colon \ \ 0 < \max_{j=1,\ldots,t} |Q_j(g\bar{x})| < \varepsilon. \tag{5}$$

Corollary 5 *Statements of Corollary 3 and Theorem 4 are valid for Δ'.*

The next section is devoted to the proofs of stated results. Some open problems are discussed in Section 3.

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2 Proofs

Proofs of Theorems 1 and 2 involve only elementary linear algebra. Theorem 1 follows from Lemmas 7 and 8 below. To prove Theorem 2, we reduce it to a question about a system of linear forms. The proof of Corollary 3(i) uses Lemmas 7 and 8 and the result of Jarník [Ja31]. Corollary 3(ii) and Theorem 4 are deduced from known properties of certain flows on the space of unimodular lattices.

If it is not stated otherwise, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

The following three lemmas are used in the proof of Theorem 1.

Lemma 6 *Let Q_j, $j = 1, \ldots, t$, be nondegenerate indefinite quadratic forms in d variables, $d \geq 3$, whose zero hypersurfaces intersect along a line ℓ, and the span of normal vectors to the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$, at a point of ℓ has dimension $d-1$. Denote by π the orthogonal projection on the line ℓ. For m, M, $\varepsilon > 0$, put*

$$\begin{aligned} R_{m,M} &= \{\bar{x} \in \mathbb{R}^d \colon \|\pi(\bar{x})\| \geq M, \|\bar{x} - \pi(\bar{x})\| \leq m\|\pi(\bar{x})\|\}, \\ T(\varepsilon) &= \{\bar{x} \in R_{m,M} \colon \|\pi(\bar{x})\| \cdot \|\bar{x} - \pi(\bar{x})\| \leq \varepsilon\}, \\ S(\varepsilon) &= \{\bar{x} \in R_{m,M} \colon |Q_j(\bar{x})| \leq \varepsilon, j = 1, \ldots, t\}. \end{aligned}$$

Then there exist $m, M, c_1, c_2 > 0$ such that for every $\varepsilon \in (0,1)$,

$$T(c_1\varepsilon) \subseteq S(\varepsilon) \subseteq T(c_2\varepsilon).$$

Proof It is enough to prove the lemma for a system of quadratic forms Q_j, $j = 1, \ldots, d-1$, such that normal vectors to $\{Q_j = 0\}$, $j = 1, \ldots, d-1$, at a point of ℓ are linearly independent.

For a vector $\bar{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, put $x = (x_1, \ldots, x_{d-1})$.

Using a change of variables, we can transform ℓ to the line $\{x_i = 0 \colon i = 1, \ldots, d-1\}$. Thus, it is enough to consider the case when the line ℓ is as above. Then

$$Q_i(\bar{x}) = q_i(x) + x_d L_i(x), \quad i = 1, \ldots, d-1. \tag{6}$$

Note that L_i is independent of x_d because $Q_i(0, \ldots, 0, 1) = 0$. The tangent plane to $\{Q_i = 0\}$ at $(0, \ldots, 0, 1)$ is $\{L_i = 0\}$.

Let

$$T(a, \varepsilon) = T(\varepsilon) \cap \{x_d = a\} \quad \text{and} \quad S(a, \varepsilon) = S(\varepsilon) \cap \{x_d = a\}.$$

For $|a| \geq M$, we have

$$T(a, \varepsilon) = \{(x, a) \colon |a| \cdot \|x\| \leq \varepsilon, \|x\| \leq m|a|\}, \tag{7}$$

$$S(a, \varepsilon) = \{(x, a) \colon |q_i(x) + aL_i(x)| < \varepsilon, i = 1, \ldots, d-1; \|x\| \leq m|a|\}. \tag{8}$$

First, we claim that for fixed sufficiently small $m > 0$,

$$\operatorname{diam}(S(a, \varepsilon)) \to 0 \tag{9}$$

uniformly on $\varepsilon \in (0,1)$ as $|a| \to \infty$. Let $L = (l_{ij})$ be a linear map where l_{ij} are the coefficients of the linear forms L_i. Note that L is nondegenerate because the normal vectors to $\{Q_j = 0\}$, $j = 1, \ldots, d-1$, at a point of ℓ are linearly independent. Using (6) and (8), we have

$$|aL_i(x)| \le \varepsilon + |q_i(x)| \le 1 + \beta_i \|x\|^2 \le 1 + \beta_i m|a| \cdot \|x\|. \tag{10}$$

for $x \in S(a, \varepsilon)$ and $i = 1, \ldots, d-1$.

Suppose that there exists a sequence $(u^{(n)}, a_n) \in S(a_n, \varepsilon)$ with $|a_n| \ge 1$, and $\|u^{(n)}\| \to \infty$. We may assume that $\frac{u^{(n)}}{\|u^{(n)}\|} \to v_0$ for some v_0 on the unit sphere S^{d-2}. From (10) we get

$$|L_i(u^{(n)})| \le \frac{1}{|a_n|} + \beta_i m \|u^{(n)}\| \le 1 + \beta_i m \|u^{(n)}\|, \quad i = 1, \ldots, d-1.$$

It follows that

$$|L_i(v_0)| \le \beta_i m, \quad i = 1, \ldots, d-1. \tag{11}$$

Take

$$m = \frac{1}{2} \min \left\{ \max \left\{ \beta_i^{-1} |L_i(x)| \colon i = 1, \ldots, d-1 \right\} : x \in S^{d-2} \right\}. \tag{12}$$

Note that $m > 0$ because L_i, $i = 1, \ldots, d-1$, are linearly independent. Then the equations (11) and (12) contradict each other. This shows that $\operatorname{diam}(S(a, \varepsilon))$ is bounded for $\varepsilon \in (0,1)$ and $a \ge 1$. From the first part of (10), we see that $|a| \cdot \|L(x)\|$ for $x \in S(a, \varepsilon)$ is bounded too. Since L is nondegenerate, for some $c > 0$, $\operatorname{diam}(S(a, \varepsilon)) < \frac{c}{|a|}$ when $\varepsilon \in (0,1)$ and $a \ge 1$. This proves the claim (9).

Consider a transformation $u \colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ defined by

$$u_i = q_i(x) + aL_i(x), \quad i = 1, \ldots, d-1.$$

By (8),

$$S(a, \varepsilon) = \{(x, a) \colon |u_i(x)| < \varepsilon, i = 1, \ldots, d-1; \|x\| \le m|a|\}. \tag{13}$$

The Jacobian of the transformation u is $J = \left(\frac{\partial q_i}{\partial x_j}\right) + aL$. Note that $\det J(0) = \det L \ne 0$. Therefore by (9), u is a diffeomorphism on $S(a, \varepsilon)$ for sufficiently large $|a|$. Let $g = dx_1^2 + \cdots + dx_{d-1}^2$ be the standard Riemann metric, $u^* g = du_1^2 + \cdots + du_{d-1}^2$ be the pull-back of g under the transformation u, and $L^* g$ be the pull-back of g under L. Clearly, for some $r_1, r_2 > 0$,

$$r_1 g < L^* g < r_2 g.$$

Since $a^{-1} J \to L$ as $|a| \to \infty$ uniformly on compact sets of $\mathbb{R}^{d-1}$, $\frac{u^* g}{a^2} \to L^* g$ uniformly on compact sets. Therefore,

$$r_1 g \le \frac{u^* g}{a^2} \le r_2 g \tag{14}$$

on $S(a, \varepsilon)$ for $|a|$ sufficiently large (say $|a| \geq M$). Note that $\|x\|$ is the distance to the origin with respect to the metric g, and $\|u(x)\|$ is the distance to the origin with respect to u^*g. Therefore, it follows from (14) that for $x \in S(a, \varepsilon)$ and $|a| \geq M$,

$$r_1^{1/2}|a| \cdot \|x\| \leq \|u(x)\| \leq r_2^{1/2}|a| \cdot \|x\|.$$

By (7) and (13), for $|a| \geq M$,

$$T\left(a, r_2^{-1/2}\varepsilon\right) \subseteq S(a, \varepsilon) \subseteq T\left(a, r_1^{-1/2}\sqrt{2}\varepsilon\right).$$

The lemma is proved. $\square$

Lemma 7 *For a line ℓ in $\mathbb{R}^d$, denote by π_ℓ the orthogonal projection on the line ℓ. Assume that zero hypersurfaces of a system of nondegenerate indefinite quadratic forms Q_j, $j = 1, \ldots, t$, in d variables intersect at a point, which is different from $\bar{0}$, and at every nonzero point of intersection, the span of the normal vectors to these hypersurfaces has dimension $d-1$. Then (2) holds iff there exist a line ℓ in the intersection of the hypersurfaces of $\{Q_j = 0\}$, $j = 1, \ldots, t$, and a sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ such that*

$$\|\pi_\ell(\bar{x}^{(n)})\| \to \infty \quad \text{and} \quad \|\pi_\ell(\bar{x}^{(n)})\| \cdot \|\bar{x}^{(n)} - \pi_\ell(\bar{x}^{(n)})\| \to 0. \tag{15}$$

Proof Let $m, M, c_1, c_2 > 0$ be as in Lemma 6.

Suppose that Q_j, $j = 1, \ldots, t$, satisfy (2). Then there exists a sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ such that

$$\varepsilon_n \stackrel{def}{=} \max\{|Q_j(\bar{x}^{(n)})| \colon j = 1, \ldots, t\} \to 0. \tag{16}$$

Passing if needed to a subsequence, we may assume that

$$\bar{y}^{(n)} \stackrel{def}{=} \frac{\bar{x}^{(n)}}{\|\bar{x}^{(n)}\|} \to \bar{y}_0$$

for some $\bar{y}_0$ on the unit sphere. Then $Q_j(\bar{y}_0) = 0$, $j = 1, \ldots, t$. Let ℓ be the line through $\bar{y}_0$ and the origin.

If $\|\bar{x}^{(n)}\| \not\to \infty$, we can take $\{\bar{x}^{(n)}\}$ to be bounded. Then for each j, $Q_j(\bar{x}^{(n)})$ admits only finitely many values for $n \geq 1$. Therefore, by (16), $Q_j(\bar{x}^{(n)}) = 0$, $j = 1, \ldots, t$, for sufficiently large n. We can replace $\bar{x}^{(n)}$ by $n\bar{x}^{(n)}$. Thus, we may assume that $\|\bar{x}^{(n)}\| \to \infty$.

Since $\bar{y}^{(n)} \to \bar{y}_0$, the sequence $\bar{x}^{(n)}$ is inside of the cone

$$\|\bar{x} - \pi_\ell(\bar{x})\| < m\|\pi_\ell(\bar{x})\|$$

for sufficiently large n. It follows that if $\|\pi_\ell(\bar{x}^{(n)})\| \not\to \infty$, then $\|\bar{x}^{(n)}\| \not\to \infty$. Therefore, $\pi_\ell(\bar{x}^{(n)}) \to \infty$. In particular, $\bar{x}^{(n)} \in R_{m,M}$ for sufficiently large n. By Lemma 6,

$$\|\pi_\ell(\bar{x}^{(n)})\| \cdot \|\bar{x}^{(n)} - \pi_\ell(\bar{x}^{(n)})\| \leq c_2\varepsilon_n \to 0.$$

Conversely, suppose that (15) is satisfied for some line ℓ in the intersection of the hypersurfaces $\{Q_j = 0\}$, $j = 1, \ldots, t$. Then for sufficiently large n, $\bar{x}^{(n)} \in R_{m,M}$. Applying Lemma 6, we get

$$|Q_j(\bar{x}^{(n)})| \le c_1^{-1} \|\pi_\ell(\bar{x}^{(n)})\| \cdot \|\bar{x}^{(n)} - \pi_\ell(\bar{x}^{(n)})\| \to 0, \;\; j = 1, \ldots, t.$$

This proves the lemma. □

Lemma 8 *Let $\bar{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$ with $v_j \neq 0$ for some $j = 1, \ldots, d$, and π is the orthogonal projection on the direction $\bar{v}$. Then the vector $\left(\frac{v_i}{v_j} \colon i \neq j\right) \in \mathbb{R}^{d-1}$ is well approximable of order 1 iff for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$,*

$$\|\pi(\bar{x}^{(n)})\| \to \infty \quad \textit{and} \quad \|\pi(\bar{x}^{(n)})\| \cdot \|\bar{x}^{(n)} - \pi(\bar{x}^{(n)})\| \to 0. \tag{17}$$

Proof Without loss of generality, $j = 1$. Put $\alpha_i = \frac{v_{i+1}}{v_1}$, $i = 1, \ldots, d-1$.

Let $L_i(\bar{x}) = \alpha_i x_1 - x_{i+1}$, $i = 1, \ldots, d-1$. Note that the planes defined by $L_i = 0$, $i = 1, \ldots, d-1$, intersect along the line in direction $\bar{v}$.

Suppose that $(\alpha_1, \ldots, \alpha_{d-1})$ is well approximable of order 1. There exists a sequence $\bar{x}^{(n)} \in \mathbb{Z}^d$ such that

$$|x_1^{(n)}| \cdot \left| L_i\left(\bar{x}^{(n)}\right)\right| \to 0, \;\; i = 1, \ldots, d-1, \tag{18}$$

with $x_1^{(n)} \neq 0$ for all n. It follows that

$$L_i(\bar{x}^{(n)}) \to 0, \;\; i = 1, \ldots, d-1. \tag{19}$$

Suppose that $|x_1^{(n)}|$ is bounded. A linear map $A \colon \mathbb{R}^d \to \mathbb{R}^d$, which is defined by

$$(A\bar{x})_i = L_i(\bar{x}), \; i = 1, \ldots, d-1, \quad \text{and} \quad (A\bar{x})_d = x_1, \tag{20}$$

is nondegenerate. Hence, by (19), $\bar{x}^{(n)}$ is bounded, and $L_i(\bar{x}^n) = 0$, $i = 1, \ldots, d-1$, for sufficiently large n. In this case, (17) holds for the sequence $n\bar{x}^{(n)}$.

If $|\bar{x}_1^{(n)}|$ is not bounded, we may assume that $|\bar{x}_1^{(n)}| \to \infty$. By (19), for sufficiently large n,

$$\|A\bar{x}^{(n)}\| = \left(\sum_{i=1}^{d-1} L_i(\bar{x}^{(n)})^2 + (x_1^{(n)})^2\right)^{1/2} \le \sqrt{d}|x_1^{(n)}|. \tag{21}$$

Since A is nondegenerate, for some $c_1 > 0$ and every $\bar{x} \in \mathbb{R}^d$,

$$\|\pi(\bar{x})\| \le c_1 \|A\bar{x}\|. \tag{22}$$

Let P be the plane through the origin orthogonal to $\bar{v}$. A map $C \colon P \to \mathbb{R}^{d-1}$ defined by

$$(C\bar{x})_i = L_i(\bar{x}), \;\; i = 1, \ldots, d-1,$$

is invertible. Therefore, for some $c_2 > 0$ and every $\bar{x} \in P$,

$$\|\bar{x} - \pi(\bar{x})\| \leq c_2 \max\{|L_i(\bar{x})| \colon i = 1, \ldots, d-1\}. \tag{23}$$

In fact, the last inequality holds for every $\bar{x} \in \mathbb{R}^d$ because it is independent of translations by vectors parallel to $\bar{v}$. Now (17) follows from (18), (21), (22), (23).

Conversely, suppose that (17) holds for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$. Note that for $\bar{x} \in \mathbb{R}^d$, $|L_i(\bar{x})|$ is up to a constant the distance from $\bar{x}$ to the plane $\{L_i = 0\}$, and $\|\bar{x} - \pi(\bar{x})\|$ is the distance from $\bar{x}$ to the line through $\bar{v}$. Therefore, for some $r_i > 0$ and every $\bar{x} \in \mathbb{R}^d$,

$$|L_i(\bar{x})| \leq r_i \|\bar{x} - \pi(\bar{x})\|, \quad i = 1, \ldots, d-1. \tag{24}$$

In particular, it follows that

$$L_i(\bar{x}^{(n)}) \to 0, \quad i = 1, \ldots, d-1. \tag{25}$$

Clearly, $x_1^{(n)} \neq 0$ for sufficiently large n.

Consider a linear map B defined by

$$(B\bar{x})_i = L_i(\bar{x}), \; i = 1, \ldots, d-1, \quad \text{and} \quad (B\bar{x})_d = \pi(\bar{x}).$$

Since B is nondegenerate, for some $r_3 > 0$ and every $\bar{x} \in \mathbb{R}^d$,

$$|x_1| \leq r_3 \|B\bar{x}\|. \tag{26}$$

By (17) and (25), for sufficiently large n,

$$\|B\bar{x}^{(n)}\| = \left(\sum_{i=1}^{d-1} L_i(\bar{x}^{(n)})^2 + \|\pi(\bar{x}^{(n)})\|^2 \right)^{1/2} \leq \sqrt{d} \|\pi(\bar{x}^{(n)})\|. \tag{27}$$

Finally, (18) follows from (17), (24), (26), and (27). Thus, the vector $(\alpha_1, \ldots, \alpha_{d-1})$ is well approximable of order 1. $\square$

Combining Lemmas 7 and 8, we deduce Theorem 1.

The proof of Corollary 3(i) uses Lemma 7, Lemma 8, and the result of Jarník on Hausdorff dimension of the set of well approximable vectors [Ja31]. It is known that the intersection of the set of well approximable vectors of order r in $\mathbb{R}^d$ with every nonempty open set has Hausdorff dimension $\frac{d+1}{r+1}$.

Proof of Corollary 3(i) Let $\mathcal{Q}$ be the projective variety in the complex projective space $\mathbb{P}^{d-1}(\mathbb{C})$ defined by $Q_j = 0$, $j = 1, \ldots, t$. Since at every point of $\mathcal{Q}(\mathbb{R})$ the rank of Jacobian $\frac{\partial(Q_1, \ldots, Q_t)}{\partial(x_1, \ldots, x_d)}$ is equal to $d-1$, it follows that every irreducible component of $\mathcal{Q}$ that has nonempty intersection with $\mathcal{Q}(\mathbb{R})$ has dimension zero. Therefore, $\mathcal{Q}(\mathbb{R})$ consists of finitely many points. This means that the zero hypersurfaces $\{Q_j = 0\}$,

$j = 1, \dots, t$, in $\mathbb{R}^d$ intersect along only finitely many lines that pass through the origin. Let $\bar{v}_s$, $s = 1, \dots, N$, be the direction vectors of these lines. Then zero hypersurfaces of Q_j^g, $j = 1, \dots, t$, intersect along vectors $g^{-1}\bar{v}_s$, $s = 1, \dots, N$. For some $g_s \in \mathrm{GL}(d, \mathbb{R})$, $\bar{v}_s = g_s \bar{e}_1$, where $e_1 = (1, 0, \dots, 0)$. We have

$$g^{-1}\bar{v}_s = \left((g^{-1}g_s)_{11}, \dots, (g^{-1}g_s)_{d1}\right), \quad s = 1, \dots, N.$$

By Lemma 7,

$$\Delta = \bigcup_{s=1}^{N} \Delta_s,$$

where Δ_s is the set of $g \in \mathrm{GL}(d, \mathbb{R})$ such that (15) holds for the line ℓ through the origin in direction $g^{-1}\bar{v}_s$. Clearly, it is enough to prove the theorem for each of the sets Δ_s separately.

Fix $s = 1, \dots, N$. Let

$$G_{s,i} = \{g \in \mathrm{GL}(d, \mathbb{R})\colon (g^{-1}g_s)_{i1} \neq 0\},$$

and $\Delta_{s,i} = \Delta_s \cap G_{s,i}$ for $i = 1, \dots, d$.

By Lemma 8, for $g \in G_{s,i}$, g belongs to $\Delta_{s,i}$ iff the vector

$$\left(\frac{(g^{-1}g_s)_{j1}}{(g^{-1}g_s)_{i1}}\colon j \neq i\right)$$

is well approximable of order 1. Consider maps $F_{s,i}\colon G_{s,i} \to \mathrm{M}(d, \mathbb{R})$, $i = 1, \dots, d$, defined by

$$\begin{aligned} F_{s,i}(g)_{j1} &= \frac{(g^{-1}g_s)_{j1}}{(g^{-1}g_s)_{i1}} \quad \text{for } \ j \neq i, \\ F_{s,i}(g)_{j_1 j_2} &= (g^{-1}g_s)_{j_1 j_2} \ \text{ for } \ (j_1, j_2) \notin \{(j, 1), \ j \neq i\}. \end{aligned}$$

Note that $F_{s,i}$ is a diffeomorphism on $G_{s,i}$, and $F_{s,i}(G_{s,i})$ is a complement of a finite union of hypersurfaces in $\mathrm{M}(d, \mathbb{R})$. Let W_i be the set of $g \in \mathrm{M}(d, \mathbb{R})$ such that the vector $(g_{j1}\colon j \neq i)$ is well-approximable of order 1. We have that for $g \in G_{s,i}$, g belongs to $\Delta_{s,i}$ iff $F_{s,i}(g)$ belongs to W_i, i.e. $\Delta_{s,i} = F_{s,i}^{-1}\left(W_i \cap F_{s,i}(G_{s,i})\right)$. Therefore,

$$\Delta_s = \bigcup_{i=1}^{d} \Delta_{s,i} = \bigcup_{i=1}^{d} F_{s,i}^{-1}\left(W_i \cap F_{s,i}(G_{s,i})\right). \tag{28}$$

The intersection of the set of well approximable vectors of order 1 in $\mathbb{R}^{d-1}$ with every nonempty open subset has Hausdorff dimension $\frac{d}{2}$. It follows that the intersection of the set W_i, $i = 1, \dots, d$, with every nonempty open subset of $\mathrm{M}(d, \mathbb{R})$ has Hausdorff dimension $d(d-1) + 1 + \frac{d}{2} = d^2 - \frac{d-2}{2}$ (see [BD, Section 3.5.5]). Since $F_{s,i}(G_{s,i})$ is open in $\mathrm{M}(d, \mathbb{R})$, the set $W_i \cap F_{s,i}(G_{s,i})$ has the same property. Therefore, by (28), the intersection of Δ_s with every nonempty open subset of $\mathrm{M}(d, \mathbb{R})$ has Hausdorff dimension $d^2 - \frac{d-2}{2}$. □

The following two lemmas are crucial for the proofs of Theorem 2 and Corollary 3(ii). The proofs of the lemmas are based on ideas of Dani [Da85, Da86].

Lemma 9 *Let L_i, $i = 1, \dots, d$, be linearly independent linear forms in d variables, $d \geq 2$. Define $a_t \in \mathrm{SL}(d, \mathbb{R})$ by*

$$\begin{aligned} L_1(a_t\bar{x}) &= e^{-t}L_1(\bar{x}), && (29)\\ L_2(a_t\bar{x}) &= e^{t}L_2(\bar{x}), && (30)\\ L_i(a_t\bar{x}) &= L_i(\bar{x}), \quad i = 3, \dots, d. \end{aligned}$$

Then the set $\{a_t\mathrm{SL}(d, \mathbb{Z})\colon t > 0\}$ is not bounded in $\mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ iff there exists a sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ such that

$$L_1(\bar{x}^{(n)})L_2(\bar{x}^{(n)}) \to 0 \ \text{ and } \ L_i(\bar{x}^{(n)}) \to 0, \quad i = 2, \dots, d. \tag{31}$$

Proof Let $\|\cdot\|$ be the norm on $\mathbb{R}^d$ defined by

$$\|\bar{x}\| = \max\{|L_i(\bar{x})|\colon i = 1, \dots, d\}, \quad \bar{x} \in \mathbb{R}^d.$$

By Mahler Compactness Criterion, the orbit $\{a_t\mathrm{SL}(d, \mathbb{Z})\colon t > 0\}$, is unbounded iff for some $t_n > 0$ and $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$, $a_{t_n}\bar{x}^{(n)} \to \bar{0}$, i.e.

$$\begin{aligned} \|a_{t_n}\bar{x}^{(n)}\| = \max\{&e^{-t_n}|L_1(\bar{x}^{(n)})|, e^{t_n}|L_2(\bar{x}^{(n)})|, \\ &|L_i(g\bar{x}^{(n)})|\colon i = 3, \dots, d\} \quad \to 0. \end{aligned} \tag{32}$$

It is clear that (32) implies (31). Conversely, suppose that (31) holds.

First, we consider the case when $L_2(\bar{x}^{(n)}) = 0$ for infinitely many n. Taking a subsequence, we may assume that $L_2(\bar{x}^{(n)}) = 0$ for all n. Then (32) holds for every sequence $t_n \to \infty$ such that

$$e^{-t_n}|L_1(\bar{x}^{(n)})| \to 0.$$

Now we may assume that $L_2(\bar{x}^{(n)}) \neq 0$. If $L_1(\bar{x}^{(n)})$ is bounded, then since the linear forms L_i, $i = 1, \dots, d$, are linearly independent, $\bar{x}^{(n)}$ is bounded. This implies that for sufficiently large n, $L_2(\bar{x}^{(n)}) = 0$. Thus, we may assume that $|L_1(\bar{x}^{(n)})| \to \infty$. Then (32) holds for the sequence t_n such that $e^{t_n} = |L_1(\bar{x}^{(n)})|^{1/2} \cdot |L_2(\bar{x}^{(n)})|^{-1/2}$. Note that $t_n > 0$ for sufficiently large n. This proves the lemma. □

Lemma 10 *Let L_i, $i = 1, \dots, d$, be linearly independent linear forms in d variables, $d \geq 2$. Then the set Φ of $g \in \mathrm{GL}(d, \mathbb{R})$ such that*

$$L_1(g\bar{x}^{(n)})L_2(g\bar{x}^{(n)}) \to 0 \ \text{ and } \ L_i(g\bar{x}^{(n)}) \to 0, \quad i = 2, \dots, d. \tag{33}$$

for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ has full Haar measure, and its complement has Hausdorff dimension d^2.

Proof Let $L_i^0(\bar{x}) = x_i$, $i = 1, \ldots, d$. For some $g_0 \in \mathrm{GL}(d, \mathbb{R})$, $L_i(g\bar{x}) = L_i^0(g_0 g\bar{x})$, $i = 1, \ldots, d$. Since multiplication by g_0 is a diffeomorphism of $\mathrm{GL}(d, \mathbb{R})$, we may assume without loss of generality that $g_0 = 1$ and $L_i = L_i^0$, $i = 1, \ldots, d$.

For every matrix $u \in \mathrm{GL}(d, \mathbb{R})$,

$$u = \mathrm{diag}(\lambda, 1, \ldots, 1) \cdot v$$

with $\lambda \in \mathbb{R}^\times$ and $v \in \mathrm{SL}(d, \mathbb{R})$. Moreover, the map

$$\mathrm{GL}(d, \mathbb{R}) \to \mathbb{R}^\times \times \mathrm{SL}(d, \mathbb{R})\colon u \mapsto (\lambda, v)$$

is a diffeomorphism. Since

$$L_1(u\bar{x}) = \lambda L_1(v\bar{x}), \quad L_i(ux) = L_i(v\bar{x}), \ \ i = 2, \ldots, d,$$

(33) holds for $g = u$ iff it holds for $g = v$. Therefore, if we show that the set $\Phi \cap \mathrm{SL}(d, \mathbb{R})$ has full measure in $\mathrm{SL}(d, \mathbb{R})$, this will imply that the set Φ has full measure in $\mathrm{GL}(d, \mathbb{R})$ too. Also by the argument as in [BD, Section 3.5.5], if the set $\mathrm{SL}(d, \mathbb{R}) - (\Phi \cap \mathrm{SL}(d, \mathbb{R}))$ has Hausdorff dimension $d^2 - 1$, then the set $\mathrm{GL}(d, \mathbb{R}) - \Phi$ has Hausdorff dimension d^2. Hence, the proof reduces to a question about $\mathrm{SL}(d, \mathbb{R})$.

By Lemma 9, (33) holds for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ iff the set

$$\{g^{-1} a_t g \mathrm{SL}(d, \mathbb{Z})\colon t > 0\}$$

is unbounded in $\mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$. This is equivalent to the set $\{a_t g \mathrm{SL}(d, \mathbb{Z})\colon t > 0\}$ being unbounded. It is known that the set of $g \in \mathrm{SL}(d, \mathbb{R})$ such that the orbit $\{a_t g \mathrm{SL}(d, \mathbb{Z})\colon t > 0\}$ is bounded has Haar measure zero (Moore Ergodicity Theorem [Zi, Theorem 2.2.6]), and its intersection with every nonempty open subset has Hausdorff dimension $d^2 - 1$ (Kleinbock, Margulis [KM96]). Therefore, the set of $g \in \mathrm{SL}(d, \mathbb{R})$ such that (33) holds for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ has full measure, and its complement in $\mathrm{SL}(d, \mathbb{R})$ has Hausdorff dimension $d^2 - 1$. This implies the lemma. $\square$

Proof of Theorem 2 With respect to the basis $\{\bar{f}_i\colon i = 1, \ldots, d\}$,

$$Q_j(\bar{y}) = q_j(y_2, \ldots, y_d) + y_1 l_j(y_2, \ldots, y_d), \ \ j = 1, \ldots, t, \tag{34}$$

where q_j is a quadratic form, and l_j is a linear form. Note that l_j is independent of y_1 because $Q_j(\bar{f}_1) = 0$. The tangent plane to $\{Q_j = 0\}$ at $\bar{f}_1$ is $\{l_j = 0\}$. Thus, $l_j = c_j l$, $j = 1, \ldots, t$, for some $c_j \in \mathbb{R} - \{0\}$ and a linear form l. Since $\bar{f}_1$ and $\bar{f}_i$, $i = 3, \ldots, d$, are in the tangent plane, $l(\bar{y}) = y_2$. Let $L_i(\bar{x})$, $i = 1, \ldots, d$, denote the coordinates of vector $\bar{x}$ with respect to the basis $\{\bar{f}_i\colon i = 1, \ldots, d\}$. Then

$$Q_j(\bar{x}) = q_j(L_2(\bar{x}), \ldots, L_d(\bar{x})) + c_j L_1(\bar{x}) L_2(\bar{x}), \ \ j = 1, \ldots, t,$$

and (29) holds. It follows that (2) holds for $(Q_j\colon j=1,\dots,t)$ provided that for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$,

$$L_1(\bar{x}^{(n)})L_2(\bar{x}^{(n)}) \to 0 \quad \text{and} \quad L_i(\bar{x}^{(n)}) \to 0,\ i=2,\dots,d. \tag{35}$$

Thus, the first statement of the theorem follows from Lemma 9.

Suppose that $Q=\sum_j \alpha_j Q_j$ is definite of rank $d-1$. Since Q is definite, it follows from (34) that $Q(\bar{y})=Q(y_2,\dots,y_d)$. Therefore, (2) implies (35) for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$, and the second statement of the theorem follows from Lemma 9. $\square$

Proof of Corollary 3(ii) It was shown in the proof of Theorem 2 that (2) holds for $(Q_j^g\colon j=1,\dots,t)$ provided that for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$,

$$L_1(g\bar{x}^{(n)})L_2(g\bar{x}^{(n)}) \to 0 \quad \text{and} \quad L_i(g\bar{x}^{(n)}) \to 0,\ i=2,\dots,d, \tag{36}$$

and when a linear combination of Q_j, $j=1,\dots,d$, is definite of rank $d-1$, $(Q_j^g\colon j=1,\dots,t)$ satisfies (2) iff (36) holds. Thus, Corollary 3(ii) follows from Lemma 9. $\square$

To prove Theorem 4(a), we use the following lemma:

Lemma 11 *Let Q_j, $j=1,\dots,t$, be a system of quadratic forms that satisfies the conditions of Theorem 4(a). Then there exists a nontrivial one-parameter unipotent subgroup of* $\mathrm{SL}(d,\mathbb{R})$ *which leaves all Q_j, $j=1,\dots,t$, invariant.*

Proof Let $Q_1|_V$ be nondegenerate. Then $\dim V \geq 3$. Write

$$\mathbb{R}^d = V \oplus W$$

where W is the orthogonal complement of V with respect to Q_1. Then the group

$$H \stackrel{def}{=} \mathrm{SO}(Q_1|_V) \oplus id_W$$

leaves Q_1 invariant. Since V is in the radical of $Q_j-\beta_j Q_1$, the forms Q_j, $j=2,\dots,t$ are invariant under H too. The group H is a noncompact semisimple Lie group. Hence, it contains a nontrivial one-parameter unipotent subgroup. This proves the lemma when $Q_1|_V$ is nondegenerate.

Now suppose that $Q_1|_V$ is degenerate, and $Q_1|_V \neq 0$. Choose a basis $\{\bar{f}_i\colon i=1,\dots,d\}$ of $\mathbb{R}^d$ such that $\bar{f}_1$ is in the radical of $Q_1|_V$, $\bar{f}_2$ is in V, $Q_1(\bar{f}_2)\neq 0$, and $\bar{f}_i$, $i=3,\dots,t$, are orthogonal to $\bar{f}_2$ with respect to Q_1. Since $\bar{f}_2$ is in the radical of $Q_j-\beta_j Q_1$, $j=2,\dots,t$, $\bar{f}_2$ is orthogonal to $\bar{f}_i$, $i=3,\dots,t$, with respect to Q_j, $j=2,\dots,t$. In the basis $\{\bar{f}_i\colon i=1,\dots,d\}$,

$$Q_j(\bar{y}) = \beta_j(y_2^2 + y_1 L(y_3,\dots,y_d)) + q_j(y_3,\dots,y_d), \quad j=1,\dots,t$$

for some linear form L and quadratic forms q_j, $j = 1, \ldots, t$. (Here we put $\beta_1 = 1$.) Note that L is the same all j because $\bar{f}_1$ is in the radical of $Q_j - \beta_j Q_1$, $j = 2, \ldots, t$. Define a linear transformation $u_t \in \mathrm{SL}(d, \mathbb{R})$ by

$$\begin{aligned} u_t &: y_1 \mapsto y_1 - 2ty_2 - t^2 L(y_3, \ldots, y_d), \\ u_t &: y_2 \mapsto y_2 + tL(y_3, \ldots, y_d), \\ u_t &: y_i \mapsto y_i, \quad i \geq 3. \end{aligned}$$

Then $\{u_t \colon t \in \mathbb{R}\}$ is a nontrivial one-parameter unipotent group that stabilizes Q_j, $j = 1, \ldots, t$.

It remains to consider the case when $Q_1|_V = 0$. Choose a basis $\{\bar{f}_i \colon i = 1, \ldots, d\}$, such that $\bar{f}_1$ and $\bar{f}_2$ is in V. With respect to this basis,

$$Q_j(\bar{y}) = \beta_j(y_1 L_1(y_3, \ldots, y_d) + y_2 L_2(y_3, \ldots, y_d)) + q_j(y_3, \ldots, y_d)$$

for some linear forms L_1, L_2, and quadratic forms q_j, $j = 1, \ldots, t$. (Here we put $\beta_1 = 1$.) Note that L_1 and L_2 are independent of j because $\bar{f}_1$ and $\bar{f}_2$ are in the radical of $Q_j - \beta_j Q_1$, $j = 2, \ldots, t$. The linear forms L_1 and L_2 are not zero because the quadratic form Q_1 is nondegenerate. Define a linear transformation $v_t \in \mathrm{SL}(d, \mathbb{R})$ by

$$\begin{aligned} v_t &: y_1 \mapsto y_1 + tL_2(y_3, \ldots, y_d), \\ v_t &: y_2 \mapsto y_2 - tL_1(y_3, \ldots, y_d), \\ v_t &: y_i \mapsto y_i, \quad i \geq 3. \end{aligned}$$

Then $\{v_t \colon t \in \mathbb{R}\}$ is a nontrivial one-parameter unipotent group that stabilizes Q_j, $j = 1, \ldots, t$. We have proved the lemma. □

The following Lemma is used in the proof of Theorem 4(b). Its proof is essentially the same as the proof of Lemma 9 and is omitted.

Lemma 12 *Let L_i, $i = 1, \ldots, d$, be linearly independent linear forms in d variables, $d \geq 2$, and a_t is defined as in (29). Then the set $\{a_t \mathrm{SL}(d, \mathbb{Z}) \colon t \in \mathbb{R}\}$ is not bounded in $\mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ iff there exists a sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ such that*

$$L_1(\bar{x}^{(n)}) L_2(\bar{x}^{(n)}) \to 0 \ \text{ and } \ L_i(\bar{x}^{(n)}) \to 0, \ \ i = 3, \ldots, d. \tag{37}$$

Proof of Theorem 4 Let $G = \mathrm{SL}(d, \mathbb{R})$ and $\Gamma = \mathrm{SL}(d, \mathbb{Z})$.

To prove (a), we show first that the complement of $\Delta \cap G$ in G is contained in a countable union of submanifolds of dimension at most $d^2 - d$. Denote by $U \subset G$ a one-parameter unipotent subgroup that stabilizes all Q_j, $j = 1, \ldots, t$. Such a subgroup exists by Lemma 11. By Ratner's topological rigidity [Ra91], the set Ω of $g \in G$ such that $Ug\Gamma$ is not dense in G/Γ is a countable union of sets of the form $F\Gamma$ where F is a connected Lie subgroup of G. It is known that the maximal dimension

of a proper connected subgroup of G is d^2-d [OV, Sec. 3.3]. By Mahler compactness criterion, for every $g \in G-\Omega$, there exist $u_n \in U$ and $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$ such that $u_n g \bar{x}^{(n)} \to \bar{0}$ as $n \to \infty$. Then

$$Q_j(g\bar{x}^{(n)}) = Q_j(u_n g \bar{x}^{(n)}) \to 0, \;\; j = 1, \ldots, t.$$

Therefore, $G - \Delta \cap G$ is contained in Ω.

Let $\mathrm{GL}^{\pm}(d,\mathbb{R})$ be the group of matrices with positive/negative discriminant. It is clear that

$$\Delta \cap \mathrm{GL}^{+}(d,\mathbb{R}) = \bigcup_{\lambda>0} \lambda \cdot (\Delta \cap G) \tag{38}$$

Therefore, the complement of $\Delta \cap \mathrm{GL}^{+}(d,\mathbb{R})$ in $\mathrm{GL}^{+}(d,\mathbb{R})$ is contained in a countable union of submanifolds of dimension at most d^2-d+1. Similarly,

$$\Delta \cap \mathrm{GL}^{-}(d,\mathbb{R}) = \bigcup_{\lambda>0} \lambda g_0 \cdot (\tilde{\Delta} \cap G) \tag{39}$$

for a fixed $g_0 \in \mathrm{GL}^{-}(d,\mathbb{R})$ and $\tilde{\Delta} \stackrel{def}{=} \Delta(Q_1^{g_0}, \ldots, Q_t^{g_0})$. It follows that the complement of $\Delta \cap \mathrm{GL}^{-}(d,\mathbb{R})$ in $\mathrm{GL}^{-}(d,\mathbb{R})$ is contained in a countable union of submanifolds of dimension at most d^2-d+1. This proves (a).

Now we prove (b). Take a basis $\{\bar{f}_i\colon i = 1, \ldots, d\}$ such that $V = \langle \bar{f}_1, \bar{f}_2 \rangle$, and $\bar{f}_i$, $i = 3, \ldots, d$ is orthogonal to V with respect to Q_1. It follows from the definition of V that $\bar{f}_i$, $i = 3, \ldots, d$ are orthogonal to V with respect to Q_j, $j = 2, \ldots, t$, too. In addition, we can change $\bar{f}_1$ and $\bar{f}_2$ such that with respect to the basis $\{\bar{f}_i\colon i = 1, \ldots, d\}$,

$$Q_j(\bar{y}) = \beta_j y_1 y_2 + q_j(y_3, \ldots, y_d), \;\; j = 1, \ldots, t \tag{40}$$

for some quadratic forms q_j, $j = 1, \ldots, t$. (Here we put $\beta_1 = 1$.) Let $Q \stackrel{def}{=} \sum_j \alpha_j Q_j$ be a definite form of rank $d-2$. It follows from (40) that $Q(\bar{y}) = Q(y_3, \ldots, y_d)$. Denote by $L_i(\bar{x})$, $i = 1, \ldots, d$, the coordinates of a vector $\bar{x}$ with respect to the basis $\{\bar{f}_i\colon i = 1, \ldots, d\}$. Then (2) holds for $(Q_j^g\colon j = 1, \ldots, t)$ with $g \in G$ iff for some sequence $\bar{x}^{(n)} \in \mathbb{Z}^d - \{\bar{0}\}$,

$$L_1(g\bar{x}^{(n)}) L_2(g\bar{x}^{(n)}) \to 0 \;\text{ and }\; L_i(g\bar{x}^{(n)}) \to 0, \;\; i = 3, \ldots, d.$$

By Lemma 12, this is equivalent to the set $\{g^{-1} a_t g \Gamma\colon t \in \mathbb{R}\}$ being unbounded in G/Γ. Therefore, by the result of Kleinbock and Margulis [KM96], the set $G-(\Delta \cap G)$ has Hausdorff dimension d^2-1.

To show that the complement of Δ in $\mathrm{GL}(d,\mathbb{R})$ has Hausdorff dimension d^2, one may use (38), (39), and [BD, Section 3.5.5]. This proves (b). $\square$

Proof of Corollary 5 Under the conditions in (i), the intersection of hypersurfaces $Q_j = 0$, $j = 1, \ldots, t$, consists of finitely many lines that pass through the origin. Denote by $\bar{v}_s$, $s = 1, \ldots, N$, the direction vectors of these lines. Note that for

$g \in \mathrm{GL}(d,\mathbb{R})$, hypersurfaces $Q_j^g = 0$, $j = 1,\dots,t$, intersect along lines in directions $g^{-1}\bar{v}_s$, $s = 1,\dots,N$. For $s = 1,\dots,N$ and $\bar{x} \in \mathbb{R}^d - \{\bar{0}\}$, denote by $R_s(\bar{x})$ the set of $g \in \mathrm{GL}(d,\mathbb{R})$ such that $g^{-1}\bar{v}_s$ is collinear with $\bar{x}$. The set $R_s(\bar{x})$ is a submanifold of dimension $d^2 - (d-1)$. Since

$$\Delta - \Delta' \subseteq \bigcup_{1\le s\le N;\ \bar{x}\in\mathbb{Z}^d-\{\bar{0}\}} R_s(\bar{x}),$$

the set $\Delta - \Delta'$ has Hausdorff dimension at most $d^2 - (d-1) < d^2 - \frac{d-2}{2}$. This shows that the set Δ' has Hausdorff dimension $d^2 - \frac{d-2}{2}$.

To prove (ii), we define

$$K_j(\bar{x}) = \{g \in \mathrm{M}(d,\mathbb{R})\colon Q_j(g\bar{x}) = 0\},\ \ j = 1,\dots,t,\ \bar{x} \in \mathbb{R}^d.$$

This is an algebraic subset of $\mathrm{M}(d,\mathbb{R})$, and for $\bar{x} \neq 0$, $K_j(\bar{x}) \neq \mathrm{M}(d,\mathbb{R})$, which implies that it has measure 0. Note that

$$\Delta - \Delta' \subseteq \bigcup_{1\le j\le t;\ \bar{x}\in\mathbb{Z}^d-\{\bar{0}\}} K_j(\bar{x}), \tag{41}$$

where the last set has measure 0 and Hausdorff dimension $< d^2$. Therefore, when Δ has complement in $\mathrm{GL}(d,\mathbb{R})$ of measure zero, so does Δ'. The statement about Hausdorff dimension follows from (41) too. This proves (ii).

We use notations from the proof of Theorem 4. If we show that $G - \Delta' \cap G$ is contained in Ω, then it is possible to finish the proof as in Theorem 4(a). There exists $\bar{x}^0 \in \mathbb{R}^d$ such that $Q_j(\bar{x}^0) \neq 0$ for every $j = 1,\dots,t$. Therefore, for every $\varepsilon > 0$, there exists $\bar{x}_\varepsilon \in \mathbb{R}^d$ such that

$$0 < |Q_j(\bar{x}_\varepsilon)| < \varepsilon,\ \ j = 1,\dots,t.$$

Write $\bar{x}_\varepsilon = g_\varepsilon \bar{e}_1$ for some $g_\varepsilon \in G$ and $e_1 = (1,0,\dots,0)$. For $g \notin \Omega$, $\overline{Ug\Gamma} = G$. Thus, there exist $u \in U$ and $\gamma \in \Gamma$ such that

$$0 < |Q_j(ug\gamma\bar{e}_1)| = |Q_j(g\gamma\bar{e}_1)| < \varepsilon,\ \ j = 1,\dots,t.$$

Hence, $g \in \Delta'$. This show that $G - \Delta' \cap G \subseteq \Omega$ and proves Theorem 4(a).

In the case (b), the intersection of the hypersurfaces $\{Q_j = 0\}$, $j = 1,\dots,t$, consists of two lines. Thus, by the same argument as in the proof of Corollary 5(i), the set $\Delta - \Delta'$ has Hausdorff dimension at most $d^2 - (d-1)$. This implies part (b) of Theorem 4. $\square$

3 Open Problems

Theorems 1 and 2 provide information about property (2) only when the dimension of the space spanned by normal vectors to the hypersurfaces $\{Q_j = 0\}$, $j = 1,\dots,t$,

is either 1 or $d-1$. Clearly, this leaves a gap for $d>3$. It is of interest to investigate property (2) assuming other conditions on the intersection of the hypersurfaces $\{Q_j=0\}$, $j=1,\dots,t$. At least, one should try to give a complete answer to the question about the magnitude of the set $\Delta(Q_1,\dots,Q_t)$, which was partially studied in Corollary 3.

Corollary 3 illustrates that in some cases, property (2) has complicated Diophantine nature. This is analogous to the situation with Oppenheim conjecture in dimension 2. In higher dimensions, the following conjecture seems plausible:

Conjecture 13 *Let $d\in\mathbb{N}$ be sufficiently large. Let (Q_1,Q_2) be a pair of real nondegenerate quadratic forms in d variables such that every linear combination $\alpha Q_1+\beta Q_2$ with $\alpha^2+\beta^2\neq 0$ is indefinite, has rank ≥ 3, and does not have all rational coefficients. Then (2) holds.*

By a theorem of P. Finsler, for $d\geq 3$ every linear combinations $\alpha Q_1+\beta Q_2$ with $\alpha^2+\beta^2\neq 0$ is indefinite iff the intersection of the hypersurfaces $\{Q_1=0\}$ and $\{Q_2=0\}$ is not equal to $\{\bar{0}\}$. (This theorem was proved independently by many authors (see [Uh79]).) Therefore, under the conditions of Conjecture 13, the set

$$\{\bar{x}\colon |Q_i(\bar{x})|<\varepsilon, i=1,2\}$$

is not compact.

A question analogous to this conjecture was studied by Dani and Margulis [DM90]. They showed that if Q is a nondegenerate indefinite quadratic form, and L is a linear form in 3 variables such that the plane $\{L=0\}$ is tangent to the hypersurface $\{Q=0\}$, and every linear combination $\alpha Q+\beta L^2$ with $\alpha^2+\beta^2\neq 0$ does not have all rational coefficients, then $\{(Q(\bar{x}),L(\bar{x}))\colon \bar{x}\in\mathbb{R}^d\}$ is dense in $\mathbb{R}^2$. Similar result holds for a pair consisting of a linear form and a quadratic form in d variables, $d\geq 4$ [Go].

Some partial results towards Conjecture 13 were obtained in [Co73] by R. J. Cook. He studied pairs of diagonal quadratic forms in d variables, $d\geq 9$, with algebraic coefficients.

Let us illustrate the conjecture by several examples:

1. Let $\mathbb{R}^d=V_1\oplus V_2$ be a direct sum of vector spaces V_1 and V_2 of dimension at least 3. Let Q_1 and Q_2 be indefinite quadratic forms such that $Q_i|_{V_i}$ is nondegenerate, $Q_i|_{V_j}=0$ for $i\neq j$, and $V_1\perp V_2$ with respect to Q_1 and Q_2. According to Conjecture 13, (2) should hold for (Q_1,Q_2) provided that every nonzero linear combination of Q_1 and Q_2 does not have all rational coefficients. This seems to be one of the easiest special cases of the conjecture. It should be possible to attack this case using the original approach of Margulis [Ma89] and Ratner's topological rigidity [Ra91].

2. Another promising case of Conjecture 13 is when a pair (Q_1,Q_2) of quadratic forms satisfies the conditions of Theorem 4(a). By Lemma 11, both Q_1 and Q_2 are invariant under a nontrivial unipotent subgroup. Hence, one can use Ratner's topological rigidity.

3. In general, the group of linear transformations that leaves both forms invariant may be finite. If this is the case, the method of Margulis, which is based on dynamics on homogeneous spaces of Lie groups, does not work. This problem can appear in every dimension even when the hypersurfaces $\{Q_1 = 0\}$ and $\{Q_2 = 0\}$ have a common tangent plane. For example, when

$$Q_1(\bar{x}) = \sum_{i=1}^{d} x_{d-i+1}x_i, \quad Q_2(\bar{x}) = \sum_{i=1}^{d-1} x_{d-i}x_i + \alpha Q_1(\bar{x}),$$

up to a linear change of variables, the group that stabilizes both Q_1 and Q_2 is finite.

References

[BD] V. I. Bernik, M. M. Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics 137, Cambridge University Press, Cambridge, 1999.

[Co73] R. J. Cook, *Simultaneous quadratic inequalities*, Acta Arith. 25, 337–346, 1973.

[Da85] S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. Reine Angew. Math. 359, 55–89, 1985.

[Da86] S. G. Dani, *Bounded orbits of flows on homogeneous spaces*, Comment. Math. Helv. 61, 636-660, 1986.

[Da00] S. G. Dani, *On Values of linear and quadratic forms at integer points*, Number theory, pp. 107–119, Trends Math., Birkhäuser, Basel, 2000.

[DM90] S. G. Dani, G. A. Margulis, *Orbit closures of generic unipotent flows on homogeneous spaces of* $\mathrm{SL}(3, R)$, Math. Ann. 286, 101–128, 1990.

[Go] A. Gorodnik, *Oppenheim conjecture for a pair consisting of a linear form and a quadratic form*, to appear in Transactions of AMS.

[Ja31] V. Jarník, *Über die simultanen diophantischen Approximationen*, Math. Z. 33, 503–543, 1931.

[KM96] D. Kleinbock, G. Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Sinaĭ's Moscow Seminar on Dynamical Systems, 141–172, Amer. Math. Soc. Transl. Ser. 2, 171, Amer. Math. Soc., Providence, RI, 1996.

[Ma89] G. Margulis, *Discrete subgroups and ergodic theory*, Number theory, trace formulas and discrete groups (Oslo, 1987), 377–398, Academic Press, Boston, MA, 1989.

[Ma97] G. Margulis, *Oppenheim conjecture*, Fields Medalists' lectures, pp. 272–327, World Sci. Publishing, River Edge, NJ, 1997.

[OV] A. Onishchik, E. Vinberg, *Lie groups and Lie algebras*, in Encyclopaedia Math. Sci., Vol. 41, Springer-Verlag, New York – Berlin, 1994.

[Ra91] M. Ratner, *Raghunathan topological conjecture and distributions of unipotent flows*, Duke Math. J. 63, 235–280, 1991.

[Uh79] F. Uhlig, *A recurring theorem about pairs of quadratic forms and extensions: a survey*, Linear Algebra Appl. 25, 219–237, 1979.

[Zi] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Basel, 1984.

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